Frobenius Curvature, Electromagnetic Strain and Description of Photon-like Objects

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Abstract

This paper aims to present a general idea for description of spatially finite physical objects with a consistent nontrivial translational-rotational dynamical structure and evolution as a whole, making use of the mathematical concepts and structures connected with the Frobenius integrability/nonintegrability theorems and electromagnetic strain quantities. The idea is based on consideration of *nonintegrable* subdistributions of some appropriate completely integrable distribution (differential system) on a manifold and then to make use of the corresponding curvatures and correspondingly directed strains as measures of interaction, i.e. of energy-momentum exchange among the physical subsystems mathematically represented by the nonintegrable subdistributions. The concept of photon-like object is introduced and description (including lagrangian) of such objects in these terms is given.

1 Introduction

At the very dawn of the 20th century Planck (Planck 1901) proposed and a little bit later Einstein (Einstien 1905) appropriately used the well known and widely used through the whole last century simple formula $E = h\nu$, $h = const > 0$. This formula marked the beginning of a new era and became a real symbol of the physical science during the following years. According to the Einstein's interpretation it gives the full energy E of *really existing* light quanta of frequency $\nu = const$, and in this way a new understanding of the nature of the electromagnetic field was introduced: the field has structure which contradicts the description given by Maxwell vacuum equations. After De Broglie's (De Broglie 1923) suggestion for the particle-wave nature of the electron obeying the same energy-frequency relation, one could read Planck's formula in the following way: *there are physical objects in Nature the very existence of which is strongly connected to some periodic (with time period $T = 1/\nu$) process of intrinsic for the object nature and such that the Lorentz invariant product ET is equal to h.* Such a reading should suggest that these objects do NOT admit point-like approximation since the relativity principle for free point particles requires straight-line uniform motion, hence, no periodicity should be allowed.

Although the great (from pragmatic point of view) achievements of the developed theoretical approach, known as *quantum theory*, the great challenge to build an adequate description of individual representatives of these objects, especially of light quanta called by Lewis *photons* (Lewis 1926) is still to be appropriately met since the efforts made in this direction, we have to admit, still have not brought satisfactory results. Recall that Einstein in his late years recognizes (Speziali 1972) that "the whole fifty years of conscious brooding have not brought me nearer to the answer to the question "what are light quanta", and now, half a century later, theoretical physics still needs progress to present a satisfactory answer to the question "what is a photon".

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We consider the corresponding theoretically directed efforts as necessary and even *urgent* in view of the growing amount of definite experimental skills in manipulation with individual photons, in particular, in connection with the experimental advancement in the "quantum computer" project. The dominating modern theoretical view on microobjects is based on the notions and concepts of quantum field theory (QFT) where the structure of the photon (as well as of any other microobject) is accounted for mainly through the so called *structural function*, and highly expensive and delicate collision experiments are planned and carried out namely in the frame of these concepts and methods (see the 'PHOTON' Conferences Proceedings, some recent review papers: Dainton 2000; Stumpf, Borne 2001; Godbole 2003; Nisius 2001). Going not in details we just note a special feature of this QFT approach: if the study of a microobject leads to conclusion that it has structure, i.e. it is not point-like, then the corresponding constituents of this structure are considered as point-like, so the point-likeness stays in the theory just in a lower level.

In this paper we follow another approach based on the assumption that the description of the available (most probably NOT arbitrary) spatial structure of photon-like objects can be made by *continuous finite/localized* functions of the three space variables. The difficulties met in this approach consist mainly, in our view, in finding adequate enough mathematical objects and solving appropriate PDE. The lack of sufficiently reliable corresponding information made us look into the problem from as general as possible point of view on the basis of those properties of photon-like objects which may be considered as most undoubtedly trustful, and in some sense, *identifying*. The analysis made suggested that such a property seems to be the fact that *the propagation of an individual photon-like object necessarily includes a straight-line translational uniform component*, so we shall focus on this property in order to see what useful for our purpose suggestions could be deduced and what appropriate structures could be constructed. All these suggestions and structures should be the building material for a step-by-step creation a *self-consistent* system. From physical point of view this should mean that the corresponding properties may combine to express a dynamical harmony in the inter-existence of appropriately defined subsystems of a finite and time stable larger physical system.

The plan of this paper is the following. In Sec.2 we introduce and comment the concept of *photon-like object*. In Sec.3 we recall some basic facts from Frobenius integrability theory, then we consider its possibilities to describe interaction between/among subsystems, mathematically represented by non-integrable subdistributions of an integrable distribution, and finally we introduce objects and structures in correspondence with the notion for photon-like object of Sec.2. In Sec.4 we make use of these objects to define corresponding relativistic strain tensor(s) and other related objects, and establish important relations with the curvature properties of the subdistributions considered in Sec.3. In Sec.5 we show explicitly how the translational-rotational consistency could be accounted for. In Sec.6 we consider possible lagrangian approaches giving appropriate equations and spatially finite solutions with photon-like properties and behavior. In the concluding Sec.6 we discuss the results obtained and present some views for further development.

2 The notion of photon-like object

We begin with the notice that any notion of a physical object must unify two kinds of properties of the object considered: *identifying* or *proper*, and *kinematical*. The identifying properties stay unchanged throughout the existence of the object, and the kinematical properties describe those changes, called *admissible*, which do NOT lead to destruction of the object. Correspondingly, physics introduces two kinds of quantities, proper and kinematical, but the more important quantities used in theoretical physics turn out to be the *dynamical* quantities which, as a rule, are functions of the proper and kinematical ones. In view of this we introduce the following notion

of Photon-like object (we shall use the abbreviation "PhLO" for "Photon-like object(s)"):

PhLO are real massless time-stable physical objects with a consistent translational-rotational dynamical structure.

We give now some explanatory comments, beginning with the term *real*. **First** we emphasize that this term means that we consider PhLO as *really* existing *physical* objects, not as appropriate and helpful but imaginary (theoretical) entities. Accordingly, PhLO **necessarily carry energy-momentum**, otherwise, they could hardly be detected. **Second**, PhLO can undoubtedly be *created* and *destroyed*, so, no point-like and infinite models are reasonable: point-like objects are assumed to have no structure, so they can not be destroyed since there is no available structure to be destroyed; creation of infinite objects (e.g. plane waves) requires infinite quantity of energy to be transformed from one kind to another for finite time-period, which seems also unreasonable. Accordingly, PhLO are *spatially finite* and have to be modeled like such ones, which is the only possibility to be consistent with their "created-destroyed" nature. It seems hardly believable that spatially infinite and indestructible physical objects may exist at all. **Third**, "spatially finite" implies that PhLO may carry only *finite values* of physical (conservative or non-conservative) quantities. In particular, the most universal physical quantity seems to be the energy-momentum, so the model must allow finite integral values of energy-momentum to be carried by the corresponding solutions. **Fourth**, "spatially finite" means also that PhLO *propagate*, i.e. they do not "move" like classical particles along trajectories, therefore, partial differential equations should be used to describe their evolution in time.

The term "**massless**" characterizes physically the way of propagation: the *integral* energy E and *integral* momentum p of a PhLO should satisfy the relation $E = cp$, where c is the speed of light in vacuum, and in relativistic terms this means that their integral energy-momentum vector *must be isotropic*, i.e. it must have zero module with respect to Lorentz-Minkowski (pseudo)metric in $\mathbb{R}^4$. If the object considered has spatial and time-stable structure, so that the translational velocity of every point where the corresponding field functions are different from zero must be equal to c, we have in fact null direction in the space-time intrinsically determined by the PhLO. Such a direction is formally defined by a null vector field $\bar{\zeta}, \bar{\zeta}^2 = 0$. The integral trajectories of this vector field are isotropic (or null) straight lines. It follows that with every PhLO a null direction is *necessarily* associated, so, canonical coordinates $(x^1, x^2, x^3, x^4) = (x, y, z, \xi = ct)$ on $\mathbb{R}^4$ may be chosen such that in the corresponding coordinate frame $\bar{\zeta}$ to have only two non-zero components of magnitude 1: $\bar{\zeta}^\mu = (0, 0, -\varepsilon, 1)$, where $\varepsilon = \pm 1$ accounts for the two directions along the coordinate z. Further such a coordinate system will be called *$\bar{zeta}$*-adapted and will be of main usage. It may be also expectable, that the corresponding energy-momentum tensor $T_{\mu\nu}$ of the model satisfies the relation $T_{\mu\nu}T^{\mu\nu} = 0$, which may be considered as a localization of the integral isotropy condition $E^2 - c^2p^2 = 0$.

The term "**translational-rotational**" means that besides translational component along $\bar{\zeta}$, the propagation necessarily demonstrates some rotational (in the general sense of this concept) component in such a way that *both components exist simultaneously and consistently.* It seems reasonable to expect that such kind of behavior should be consistent only with some distinguished spatial shapes. Moreover, if the Planck relation $E = h\nu$ must be respected throughout the evolution, the rotational component of propagation should have *time-periodical* nature with time period $T = E/\nu = const$, and one of the two possible, *left* or *right*, orientations. It seems reasonable also to expect periodicity in the spatial shape of PhLO, which somehow to be related to the time periodicity.

The term "**dynamical structure**" means that the propagation is supposed to be necessarily accompanied by an *internal energy-momentum redistribution*, which may be considered in the model as energy-momentum exchange between (or among) some appropriately defined subsystems. It could also mean that PhLO live in a dynamical harmony with the outside world, i.e.

any outside directed energy-momentum flow should be accompanied by a parallel inside directed energy-momentum flow.

Finally, note that if the time periodicity and the spatial periodicity should be consistently related somehow, the simplest such consistency would seem like this: the spatial size along the translational component of propagation $4l_o$ is equal to cT: $4l_o = cT$, where l_o is some finite positive characteristic constant of the corresponding solution. This would mean that every individual PhLO determines its own length/time scale.

We are going now to formulate shortly the basic idea inside which this study will be carried out.

3 Frobenius curvature and interaction

Any physical system with a dynamical structure is characterized with some internal energy-momentum redistributions, i.e. energy-momentum fluxes, during evolution. Any system of energy-momentum fluxes (as well as fluxes of other interesting for the case physical quantities subject to change during evolution, but we limit ourselves just to energy-momentum fluxes here) can be considered mathematically as generated by some system of vector fields. A *consistent* and *interelated time-stable* system of energy-momentum fluxes can be considered to correspond directly or indirectly to an integrable distribution Δ of vector fields according to the principle *local object generates integral object.* It seems reasonable to assume the following geometrization of the concept of physical interaction: *two distributions Δ_1 and Δ_2 on a manifold will be said to interact geometrically if at least one of the corresponding two curvature forms Ω_1/Ω_2 takes values, or generates objects taking values, respectively in Δ_2/Δ_1.*

The above concept of *geometrical interaction* is motivated by the fact that, in general, an integrable distribution Δ may contain various *nonintegrable* subdistributions $\Delta_1, \Delta_2, \dots$ which subdistributions may be interpreted physically as interacting subsytems. Any physical interaction between 2 subsystems is necessarily accompanied with available energy-momentum exchange between them, this could be understood mathematically as nonintegrability of each of the two subdistributions of Δ and could be naturally measured directly or indirectly by the corresponding curvatures. For example, if Δ is an integrable 3-dimensional distribution spent by the vector fields (X_1, X_2, X_3) then we may have, in general, three non-integrable, i.e. geometrically interacting, 2-dimensional subdistributions $(X_1, X_2), (X_1, X_3), (X_2, X_3)$. Finally, some interaction with the outside world can be described by curvatures of distributions (and their subdistributions) in which elements from Δ and vector fields outside Δ are involved (such processes will not be considered in this paper).

To make the above statements mathematically clearer we recall the Frobenius theorem on a manifold M^n [Godbillon 1969] (further all manifolds are assumed smooth and finite dimensional and all objects defined on M^n are also assumed smooth). If the system of vector fields $\Delta = [X_1(x), X_2(x), \dots, X_p(x)]$, $x \in M$, $1 < p < n$, satisfies $X_1(x) \wedge X_2(x) \wedge \dots, \wedge X_p(x) \neq 0$, $x \in M$ then Δ is integrable iff all Lie brackets $[X_i, X_j]$, $i, j = 1, 2, \dots, p$ are representable linearly through the very $X_i, i = 1, 2, \dots, p : [X_i, X_j] = C_{ij}^k X_k$, where C_{ij}^k are functions. Clearly, an easy way to find out if a distribution is integrable is to check if the exterior products

$$[X_i, X_j] \wedge X_1(x) \wedge X_2(x) \wedge \dots, \wedge X_p(x),\ x \in M; \quad i, j = 1, 2, \dots, p$$

are identically zero. If this is not the case (which means that at least one such Lie bracket "sticks out" of the distribution Δ) then the corresponding coefficients, which are multilinear combinations of the components of the vector fields and their derivatives, represent the corresponding curvatures. We note finally that if two subdistributions contain at least one common vector field it seems naturally to expect interaction.

In the dual formulation of Frobenius theorem in terms of differential 1-forms (i.e. Pfaff forms) we look for $(n-p)$-Pfaff forms $(\alpha^1, \alpha^2, \dots, \alpha^{n-p})$, i.e. a $(n-p)$-codistribution Δ^*, such that $\langle \alpha^m, X_j \rangle = 0$, and $\alpha^1 \wedge \alpha^2 \wedge \dots \wedge \alpha^{n-p} \neq 0$, $m = 1, 2, \dots, n-p$, $j = 1, 2, \dots, p$. Then the integrability of the distribution Δ is equivalent to the requirements

$$\mathbf{d}\alpha^m \wedge \alpha^1 \wedge \alpha^2 \wedge \dots \wedge \alpha^{n-p} = 0, \quad m = 1, 2, \dots, (n-p),$$

where $\mathbf{d}$ is the exterior derivative.

Since the idea of curvature associated with, for example, an arbitrary 2-dimensional distribution (X, Y) is to find out if the Lie bracket $[X, Y]$ has components along vector fields outside the 2-plane defined by (X, Y), in our case we have to evaluate the quantities $\langle \alpha^m, [X, Y] \rangle$, where all linearly independent 1-forms α^m annihilate $(X, Y) : \langle \alpha^m, X \rangle = \langle \alpha^m, Y \rangle = 0$. In view of the formula

$$\mathbf{d}\alpha^m(X, Y) = X(\langle \alpha^m, Y \rangle) - Y(\langle \alpha^m, X \rangle) - \langle \alpha^m, [X, Y] \rangle = -\langle \alpha^m, [X, Y] \rangle$$

we may introduce explicitly the curvature 2-form for the distribution $\Delta(X) = (X_1, \dots, X_p)$. In fact, if $\Delta(Y) = (Y_1, \dots, Y_{n-p})$ define a distribution which is complimentary to $\Delta(X)$ and $\langle \alpha^m, Y_n \rangle = \delta^m_n$, i.e. $(Y_1, \dots, Y_{n-p})$ and $(\alpha^1, \dots, \alpha^{n-p})$ are dual bases, then the corresponding curvature 2-form $\Omega_{\Delta(X)}$ should be defined by

$$\Omega_{\Delta(X)} = -\mathbf{d}\alpha^m \otimes Y_m, \ \text{ since } \ \Omega_{\Delta(X)}(X_i, X_j) = -\mathbf{d}\alpha^m(X_i, X_j) Y_m = \langle \alpha^m, [X_i, X_j] \rangle Y_m,$$

where it is meant here that $\Omega_{\Delta(X)}$ is restricted to the distribution $(X_1, \dots, X_p)$. Hence, if we call the distribution $(X_1, \dots, X_p)$ *horizontal*, and the complimentary distribution $(Y_1, \dots, Y_{n-p})$ *vertical* then the curvature 2-form acquires the status of *vertical bundle valued 2-form*. We see that the curvature 2-form distinguishes those couples of vector fields inside $\Delta(X)$ the Lie brackets of which define outside $\Delta(X)$ directed flows, and so, do not allowing to find integral manifold of $\Delta(X)$. Clearly, the supposition here for dimensional complementarity of the two distributions $\Delta(X)$ and $\Delta(Y)$ is not essential for the idea of curvature.

Hence, from physical point of view, if we make use of the quantities $\Omega_{\Delta(X)}(X_i, X_j)$ to build the components of the energy-momentum locally transferred from the system $\Delta(X)$ to the system $\Delta(Y)$, then, naturally, we have to make use of the quantities $\Omega_{\Delta(Y)}(Y_m, Y_n)$ to build the components of the energy-momentum transferred from $\Delta(Y)$ to $\Delta(X)$. It deserves to note that it is possible a dynamical equilibrium between the two systems $\Delta(Y)$ and $\Delta(X)$ to exist: each system to gain as much energy-momentum as it loses, and this to take place at every space-time point. On the other hand, the restriction of $\Omega_{\Delta(X)} = -\mathbf{d}\alpha^m \otimes Y_m, m = 1, \dots, n-p$, to the system $\Delta(Y)$, i.e. the quantities $\Omega_{\Delta(X)}(Y_m, Y_n)$, and the restriction of $\Omega_{\Delta(Y)} = -\mathbf{d}\beta^i \otimes X_i, i = 1, \dots, p, \langle \beta^i, X_j \rangle = \delta^i_j, \beta^1 \wedge \dots \wedge \beta^p \neq 0, \langle \beta^m, Y_i \rangle = 0$, to $\Delta(X)$, i.e. the quantities $\Omega_{\Delta(Y)}(X_i, X_j)$, acquire the sense of objects causing local change of the corresponding energy-momentum, i.e. differences between energy-momentum gains and losses. Therefore, if $W_{(X,Y)}$ denotes the energy-momentum transferred from $\Delta(X)$ to $\Delta(Y)$, $W_{(Y,X)}$ denotes the energy-momentum transferred from $\Delta(Y)$ to $\Delta(X)$, and $\delta W_{(X)}$ and $\delta W_{(Y)}$ denote respectively the energy-momentum changes of the two systems $\Delta(X)$ and $\Delta(Y)$, then according to the local energy-momentum conservation law we can write

$$\delta W_{(X)} = W_{(Y,X)} + W_{(X,Y)}, \quad \delta W_{(Y)} = -(W_{(X,Y)} + W_{(Y,X)}) = -\delta W_{(X)}.$$

For the case of dynamical equilibrium we have $\delta W_{(X)} = \delta W_{(Y)} = 0$, so we obtain

$$\delta W_{(X)} = 0, \quad \delta W_{(Y)} = 0, \quad W_{(Y,X)} + W_{(X,Y)} = 0.$$

As for how to build explicitly the corresponding representatives of the energy-momentum fluxes, probably, universal procedure can not be offered since the adequate mathematical representative

of the system under consideration depends strongly on the very system. If, for example, the mathematical representative is a differential form G, then the most simple procedure seems to be to "project" the curvature components $\Omega_{\Delta(X)}(X_i, X_j)$ and $\Omega_{\Delta(Y)}(Y_m, Y_n)$, as well as the components $\Omega_{\Delta(X)}(Y_i, Y_j)$ and $\Omega_{\Delta(Y)}(X_m, X_n)$ on G i.e. to consider the corresponding interior products. In the general case, appropriate quantities constructed out of the members of the introduced distributions and codistributions must be found.

Finally we note that, as we shall see further, a PhLO may be considered to represent an example of a system, functioning through a dynamical equilibrium between two appropriately defined and interacting subsystems.

We are going now to make use of the above general consideration to find appropriate objects and relations in an attempt to describe PhLO's dynamical structure and evolution in these terms.

4 PhLO dynamical structure in terms of Frobenius curvature

We consider the Minkowski space-time $M = (\mathbb{R}^4, \eta)$ with signature $sign(\eta) = (-,-,-,+)$ related to the standard global coordinates

$$(x^1, x^2, x^3, x^4) = (x, y, z, \xi = ct),$$

and the natural volume form $\omega_o = \sqrt{|\eta|} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 = dx \wedge dy \wedge dz \wedge d\xi$.

We introduce the null vector field $\bar{\zeta}$, $\bar{\zeta}^2 = 0$, which in the $\bar{\zeta}$-adapted coordinates (throughout used further) is assumed to look as follows:

$$\bar{\zeta} = -\varepsilon \frac{\partial}{\partial z} + \frac{\partial}{\partial \xi}, \quad \varepsilon = \pm 1. \tag{1}$$

Let's denote the corresponding to $\bar{\zeta}$ completely integrable 3-dimensional Pfaff system by $\Delta^*(\bar{\zeta})$. Thus, $\Delta^*(\bar{\zeta})$ is generated by three linearly independent 1-forms $(\alpha_1, \alpha_2, \alpha_3)$ which annihilate $\bar{\zeta}$, i.e.

$$\alpha_1(\bar{\zeta}) = \alpha_2(\bar{\zeta}) = \alpha_3(\bar{\zeta}) = 0; \quad \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \neq 0.$$

Instead of $(\alpha_1, \alpha_2, \alpha_3)$ we introduce the notation (A, A^*, ζ) and define ζ by

$$\zeta = \varepsilon dz + d\xi, \tag{2}$$

Now, since ζ defines 1-dimensional completely integrable Pfaff system we have the corresponding completely integrable distribution $(\bar{A}, \bar{A}^*, \bar{\zeta})$. We shall restrict our further study on PhLO of electromagnetic nature according to the following

Definition: We shall call a PhLO *electromagnetic* if it satisfies the following conditions ($\langle , \rangle$ is the coupling between forms and vectors):

1. $\langle \zeta, \bar{A} \rangle = \langle \zeta, \bar{A}^* \rangle = 0$,
2. the vector fields $(\bar{A}, \bar{A}^*)$ have no components along $\bar{\zeta}$,
3. $(\bar{A}, \bar{A}^*)$ are η-corresponding to (A, A^*) respectively .
4. $\langle A, \bar{A}^* \rangle = 0, \quad \langle A, \bar{A} \rangle = \langle A^*, \bar{A}^* \rangle$.

Remark. These relations formalize knowledge from Classical electrodynamics. In fact, our vector fields $(\bar{A}, \bar{A}^*)$ are meant to represent the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ components of a free time-dependent electromagnetic field, where, as is well known [Synge,1958], the translational propagation of the field energy-momentum along a fixed null direction with the velocity "c" is possible only if the two invariants $I_1 = \mathbf{B}^2 - \mathbf{E}^2$ and $I_2 = 2\mathbf{E}.\mathbf{B}$ are zero, because only in such

a case the energy-momentum tensor has *unique* null eigen direction. So it seems naturally to consider this property as *intrinsic* for the field and to choose it as a starting point. Moreover, in such a case the relation $(I_1)^2+(I_2)^2=0$ is equivalent to $\mathbf{E}^2+\mathbf{B}^2=2|\mathbf{E}\times\mathbf{B}|$ and this relation shows that this is the only case when a nonzero field momentum can not be made equal to zero by means of frame change. Together with the fact that the spatial direction of translational energy-momentum propagation is determined by $\mathbf{E}\times\mathbf{B}$, this motivates to introduce the vector field $\bar{\zeta}$ in this form and to assume the properties 1-4 in the above definition.

From the above conditions it follows that in the $\bar{\zeta}$-adapted coordinate system we have

$$A=u\,dx+p\,dy,\quad A^*=-\varepsilon\,p\,dx+\varepsilon\,u\,dy;\quad \bar{A}=-u\,\frac{\partial}{\partial x}-p\,\frac{\partial}{\partial y},\quad \bar{A}^*=\varepsilon\,p\,\frac{\partial}{\partial x}-\varepsilon\,u\,\frac{\partial}{\partial y},$$

where $\varepsilon=\pm 1$, and (u,p) are two smooth functions on M.

The completely integrable 3-dimensional Pfaff system (A,A^*,ζ) contains three 2-dimensional subsystems: $(A,A^*),(A,\zeta)$ and (A^*,ζ). We have the following

Proposition 1. The following relations hold:

$$\mathbf{d}A\wedge A\wedge A^*=0;\quad \mathbf{d}A^*\wedge A^*\wedge A=0;$$

$$\mathbf{d}A\wedge A\wedge\zeta=\varepsilon\big[u(p_\xi-\varepsilon p_z)-p(u_\xi-\varepsilon u_z)\big]\omega_o;$$

$$\mathbf{d}A^*\wedge A^*\wedge\zeta=\varepsilon\big[u(p_\xi-\varepsilon p_z)-p(u_\xi-\varepsilon u_z)\big]\omega_o.$$

Proof. Immediately verified.

These relations say that the 2-dimensional Pfaff system (A,A^*) is completely integrable for any choice of the two functions (u,p), while the two 2-dimensional Pfaff systems (A,ζ) and (A^*,ζ) are NOT completely integrable in general, and the same curvature factor

$$\mathbf{R}=u(p_\xi-\varepsilon p_z)-p(u_\xi-\varepsilon u_z)$$

determines their nonintegrability.

Correspondingly, the 3-dimensional completely integrable distribution (or differential system) $\Delta(\bar{\zeta})$ contains three 2-dimensional subsystems: $(\bar{A},\bar{A}^*)$, $(\bar{A},\bar{\zeta})$ and $(\bar{A}^*,\bar{\zeta})$. We have the

Proposition 2. The following relations hold (recall that $[X,Y]$ denotes the Lie bracket):

$$[\bar{A},\bar{A}^*]\wedge\bar{A}\wedge\bar{A}^*=0, \tag{3}$$

$$[\bar{A},\bar{\zeta}]=(u_\xi-\varepsilon u_z)\frac{\partial}{\partial x}+(p_\xi-\varepsilon p_z)\frac{\partial}{\partial y}, \tag{4}$$

$$[\bar{A}^*,\bar{\zeta}]=-\varepsilon(p_\xi-\varepsilon p_z)\frac{\partial}{\partial x}+\varepsilon(u_\xi-\varepsilon u_z)\frac{\partial}{\partial y}. \tag{5}$$

Proof. Immediately verified.

From these last relations (3)-(5) it follows that the distribution $(\bar{A},\bar{A}^*)$ is integrable, and it can be easily shown that the two distributions $(\bar{A},\bar{\zeta})$ and $(\bar{A}^*,\bar{\zeta})$ would be completely integrable only if the same curvature factor

$$\mathbf{R}=u(p_\xi-\varepsilon p_z)-p(u_\xi-\varepsilon u_z) \tag{6}$$

is zero (the elementary proof is omitted).

As it should be, the two projections

$$\langle A, [\bar{A}^*, \bar{\zeta}]\rangle = -\langle A^*, [\bar{A}, \bar{\zeta}]\rangle = \varepsilon u(p_\xi - \varepsilon p_z) - \varepsilon p(u_\xi - \varepsilon u_z) = -\varepsilon\,\mathbf{R}$$

are nonzero and give (up to a sign) the same factor $\mathbf{R}$. The same curvature factor appears, of course, as coefficient in the exterior products $[\bar{A}^*, \bar{\zeta}] \wedge \bar{A}^* \wedge \bar{\zeta}$ and $[\bar{A}, \bar{\zeta}] \wedge \bar{A} \wedge \bar{\zeta}$. In fact, we obtain

$$[\bar{A}^*, \bar{\zeta}] \wedge \bar{A}^* \wedge \bar{\zeta} = -[\bar{A}, \bar{\zeta}] \wedge \bar{A} \wedge \bar{\zeta} = -\varepsilon\mathbf{R}\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \mathbf{R}\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial \xi}.$$

On the other hand, for the other two projections we obtain

$$\langle A, [\bar{A}, \bar{\zeta}]\rangle = \langle A^*, [\bar{A}^*, \bar{\zeta}]\rangle = \frac{1}{2}\big[(u^2 + p^2)_\xi - \varepsilon(u^2 + p^2)_z\big]. \tag{7}$$

Clearly, the last relation (7) may be put in terms of the Lie derivative $L_{\bar{\zeta}}$ as

$$\frac{1}{2}L_{\bar{\zeta}}(u^2 + p^2) = -\frac{1}{2}L_{\bar{\zeta}}\langle A, \bar{A}\rangle = -\langle A, L_{\bar{\zeta}}\bar{A}\rangle = -\langle A^*, L_{\bar{\zeta}}\bar{A}^*\rangle.$$

Remark. Further in the paper we shall denote $\sqrt{u^2 + p^2} \equiv \phi$, and shall assume that ϕ is a *spatially finite* function, so, u and p must also be spatially finite.

Proposition 3. There is a function $\psi(u, p)$ such, that

$$L_{\bar{\zeta}}\psi = \frac{u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)}{\phi^2} = \frac{\mathbf{R}}{\phi^2}.$$

Proof. It is immediately verified that $\psi = \arctan\frac{p}{u}$ is such one.

We note that the function ψ has a natural interpretation of *phase* because of the easily verified now relations $u = \phi\cos\psi$, $p = \phi\sin\psi$, and ϕ acquires the status of *amplitude.* Since the transformation $(u, p) \rightarrow (\phi, \psi)$ is non-degenerate this allows to work with the two functions (ϕ, ψ) instead of (u, p).

From **Prop.3** we have

$$\mathbf{R} = \phi^2 L_{\bar{\zeta}}\psi = \ \phi^2(\psi_\xi - \varepsilon\psi_z)\,. \tag{8}$$

This last formula (8) shows something very important: at any $\phi \neq 0$ the curvature $\mathbf{R}$ will NOT be zero only if $L_{\bar{\zeta}}\psi \neq 0$, which admits in principle availability of rotation. In fact, lack of rotation would mean that ϕ and ψ are running plane waves along $\bar{\zeta}$. The relation $L_{\bar{\zeta}}\psi \neq 0$ means, however, that rotational properties are possible in general, and some of these properties are carried by the phase ψ. It follows that in such a case the translational component of propagation along $\bar{\zeta}$ (which is supposed to be available) must be determined essentially, and most probably entirely, by ϕ. In particular, we could expect the relation $L_{\bar{\zeta}}\phi = 0$ to hold, and if this happens, then the rotational component of propagation will be represented entirely by the phase ψ, and, more specially, by the curvature factor $\mathbf{R} \neq 0$, so, since the objects we are going to describe have consistent translational-rotational dynamical structure, further we assume that, in general, $L_{\bar{\zeta}}\psi \neq 0$.

We are going now to represent some relations, analogical to the energy-momentum relations in classical electrodynamics, determined by some 2-form F, in terms of the Frobenius curvatures given above.

The two nonintegrable Pfaff systems (A, ζ) and (A^*, ζ) define naturally corresponding 2-forms:

$$G = A \wedge \zeta \ \text{ and } \ G^* = A^* \wedge \zeta.$$

We have also the 2-vectors

$$\bar{G} = \bar{A} \wedge \bar{\zeta}, \quad \text{and} \quad \bar{G^*} = \bar{A}^* \wedge \bar{\zeta}.$$

Making use now of the Hodge $*$-operator, defined by η through the relation $\alpha \wedge \beta = \eta(*\alpha, \beta)\omega_o$, where α and β are p and $(4-p)$-forms on M, we can verify the relation: $G^* = *G$. The 2-forms G and G^* define the 2-tensor, called stress-energy-momentum tensor T_μ^ν, according to the rule

$$T_\mu^\nu = -\frac{1}{2}\big[G_{\mu\sigma}G^{\nu\sigma} + (G^*)_{\mu\sigma}(G^*)^{\nu\sigma}\big],$$

and the divergence of this tensor field can be represented in the form

$$\nabla_\nu T_\mu^\nu = \big[i(\bar{G})\mathbf{d}G\big]_\mu + \big[i(\bar{G^*})\mathbf{d}G^*\big]_\mu,$$

where $\bar{G}$ and $\bar{G^*}$ coincide with the metric-corresponding contravarint tensor fields, and $i(\bar{G}) = i(\bar{\zeta}) \circ i(\bar{A})$, $i(\bar{G^*}) = i(\bar{\zeta}) \circ i(\bar{A}^*)$, $i(X)$ is the standard insertion operator in the exterior algebra of differential forms on $\mathbb{R}^4$ defined by the vector field X. So, we shall need the quantities

$$i(\bar{G})\mathbf{d}G, \quad i(\bar{G^*})\mathbf{d}G^*, \quad i(\bar{G^*})\mathbf{d}G, \quad i(\bar{G})\mathbf{d}G^*.$$

Having in view the explicit expressions for $A, A^*, \zeta, \bar{A}, \bar{A}^*$ and $\bar{\zeta}$ we obtain

$$i(\bar{G})\mathbf{d}G = i(\bar{G^*})\mathbf{d}G^* = \frac{1}{2}L_{\bar{\zeta}}\left(\phi^2\right).\zeta\ . \tag{9}$$

Also, we obtain

$$i(\bar{G^*})\mathbf{d}G = -i(\bar{G})\mathbf{d}G^* =$$

$$= \Big[u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)\Big]dz + \varepsilon\Big[u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)\Big]d\xi = \varepsilon\mathbf{R}\,\zeta. \tag{10}$$

In the following formulae we must keep in mind the relations $\mathbf{d}\zeta = 0, \langle A, \bar{A}^*\rangle = \langle \zeta, \bar{A}^*\rangle = \langle \zeta, \bar{A}\rangle = 0$.

The two distributions $(\bar{A}, \bar{\zeta})$ and $(\bar{A}^*, \bar{\zeta})$ determine corresponding curvature forms Ω and Ω^* according to

$$\Omega = -\frac{1}{\phi^2}\mathbf{d}A^* \otimes \bar{A}^*, \quad \Omega^* = -\frac{1}{\phi^2}\mathbf{d}A \otimes \bar{A}.$$

Denoting $Z_\Omega \equiv \Omega(\bar{A}, \bar{\zeta})$, $Z^*_\Omega \equiv \Omega(\bar{A}^*, \bar{\zeta})$, $Z_{\Omega^*} \equiv \Omega^*(\bar{A}, \bar{\zeta})$ and $Z^*_{\Omega^*} \equiv \Omega^*(\bar{A}^*, \bar{\zeta})$ we obtain

$$Z_\Omega = -\frac{\varepsilon\mathbf{R}}{\phi^2}\bar{A}^*, \quad Z^*_\Omega = \frac{\bar{A}^*}{2\phi^2}L_{\bar{\zeta}}(\phi^2), \quad Z_{\Omega^*} = \frac{\bar{A}}{2\phi^2}L_{\bar{\zeta}}(\phi^2), \quad Z^*_{\Omega^*} = -\frac{\varepsilon\mathbf{R}}{\phi^2}\bar{A}. \tag{11}$$

The following relations express the connection between the curvatures and the energy-momentum characteristics.

$$i(Z_\Omega)(A \wedge \zeta) = 0, \quad i(Z_\Omega)(A^* \wedge \zeta) = \varepsilon\mathbf{R}.\zeta = -i(\bar{G})\mathbf{d}G^* = i(\bar{G^*})\mathbf{d}G, \tag{12}$$

$$i(Z_{\Omega^*})(A^* \wedge \zeta) = 0, \quad i(Z_{\Omega^*})(A \wedge \zeta) = \varepsilon\mathbf{R}.\zeta = -i(\bar{G})\mathbf{d}G^* = i(\bar{G^*})\mathbf{d}G, \tag{13}$$

$$i(Z^*_\Omega)(A \wedge \zeta) = 0, \quad i(Z^*_\Omega)(A^* \wedge \zeta) = -\frac{1}{2}L_{\bar{\zeta}}(\phi^2).\zeta = -i(\bar{G})\mathbf{d}G = -i(\bar{G^*})\mathbf{d}G^*, \tag{14}$$

$$i(Z^*_{\Omega^*})(A^* \wedge \zeta) = 0, \quad i(Z^*_{\Omega^*})(A \wedge \zeta) = -\frac{1}{2}L_{\bar{\zeta}}(\phi^2).\zeta = -i(\bar{G})\mathbf{d}G = -i(\bar{G^*})\mathbf{d}G^*. \tag{15}$$

It follows from these relations that in case of dynamical equilibrium we shall have

$$i(\bar{G})\mathbf{d}G = 0, \quad i(\bar{G^*})\mathbf{d}G^* = 0, \quad i(\bar{G^*})\mathbf{d}G + i(\bar{G})\mathbf{d}G^* = 0.$$

Resuming, we can say that Frobenius integrability viewpoint suggests to make use of one completely integrable 3-dimensional distribution (resp. Pfaff system) consisting of one isotropic and two space-like vector fields (resp. 1-forms), such that the corresponding 2-dimensional spatial subdistribution $(\bar{A}, \bar{A}^*)$ (resp. Pfaff system (A, A^*)) defines a completely integrable system, and the rest two 2-dimensional subdistributions $(\bar{A}, \bar{\zeta})$ and $(\bar{A}^*, \bar{\zeta})$ (resp. Pfaff systems (A, ζ) and (A^*, ζ)) are NON-integrable in general and give the same curvature. This curvature may be used to build quantities, physically interpreted as energy-momentum internal exchanges between the corresponding two subsystems $(\bar{A}, \bar{\zeta})$ and $(\bar{A}^*, \bar{\zeta})$ (resp.(A, ζ) and (A^*, ζ)). Moreover, rotational component of propagation will be available only if the curvature $\mathbf{R}$ is nonzero, i.e. only if an internal energy-momentum exchange takes place. We see that all physically important characteristics and relations, describing the translational and rotational components of propagation, can be expressed in terms of the corresponding Frobenius curvature. We'll see that this holds also for some integral characteristics of PhLO.

5 The electromagnetic strain viewpoint

The concept of *strain* is introduced in studying elastic materials subject to external forces of different nature: mechanical, electromagnetic, etc. In nonrelativistic continuum physics the local representatives of the external forces in this context are usually called *stresses*. Since the force means energy-momentum transfer leading to corresponding mutual energy-momentum change of the interacting objects, then according to the energy-momentum conservation law the material must react somehow to the external interference in accordance with its structure and reaction abilities. The classical strain describes mainly the abilities of the material to bear force-action from outside through deformation, i.e. through changing its shape, or, configuration. The term *elastic* now means that any two allowed configurations can be deformed to each other without appearence of holes and breakings, in particular, if the material considered has deformed from configuration C_1 to configuration C_2 it is able to return smoothly to its configuration C_1.

The general geometrical description [Marsden 1994] starts with the assumption that an elastic material is a continuum $\mathbb{B} \subset \mathbb{R}^3$ which may smoothly deform inside the space $\mathbb{R}^3$, so, it can be endowed with differentiable structure, i.e. having an elastic material is formally equivalent to have a smooth real 3-dimensional submanifold $\mathbb{B} \subset \mathbb{R}^3$. The deformations are considered as smooth maps (mostly embeddings) $\varphi : \mathbb{B} \rightarrow \mathbb{R}^3$. The spaces $\mathbb{B}$ and $\mathbb{R}^3$ are endowed with riemannian metrics $\mathbf{G}$ and g respectively (and corresponding riemannian co-metrics $\mathbf{G}^{-1}$ and g^{-1}), and induced isomorphisms $\tilde{\mathbf{G}}$ and $\tilde{g}$ between the corresponding tangent and cotangent spaces . This allows to define linear map inside every tangent space of $\mathbb{B}$ in the following way: a tangent vector $V \in T_x\mathbb{B}$, $x \in \mathbb{B}$, is sent through the differential $d\varphi$ of φ to $(d\varphi)_x(V) \in T_{\varphi(x)}\mathbb{R}^3$, then by means of the isomorphism $\tilde{g}$ we determine the corresponding 1-form (i.e. we "lower the index"), this 1-form is sent to the dual space $T^*_x\mathbb{B}$ of $T_x\mathbb{B}$ by means of the dual linear map $(d\varphi)^* : T^*_{\varphi(x)}\mathbb{R}^3 \rightarrow T^*_x\mathbb{B}$, and finally, we determine the corresponding tangent vector by means of the isomorphism $\tilde{\mathbf{G}}^{-1}$ (i.e. we "raise the index" correspondingly). The so obtained linear map

$$\mathbf{C}_x := \left[\tilde{\mathbf{G}}^{-1} \circ (d\varphi)^* \circ \tilde{g} \circ (d\varphi)\right]_x : T_x\mathbb{B} \rightarrow T_x\mathbb{B}$$

(which is denoted in [Marsden 1994] by $(\mathbf{F^T F})_x$), extended to the whole $\mathbb{B}$, is called *Caushy-Green deformation tensor field*. Now, the combination

$$\mathbf{E}_x := \frac{1}{2}\left[(\tilde{\mathbf{G}} \circ \mathbf{C} - \mathbf{G})\right]_x = \frac{1}{2}\left[(d\varphi)^* \circ \tilde{g} \circ (d\varphi) - \mathbf{G}\right]_x : T_x\mathbb{B} \times T_x\mathbb{B} \rightarrow \mathbb{R}$$

is called *Lagrangian strain tensor field*. Following corresponding linearization procedure (Ch.4 in [Marsden 1994]) defined by an appropriate vector field X on $\mathbb{B}$, representing an infinitesimal

displacement of $\mathbb{B}$, it can be shown that the linearization of $\mathbf{E}$ reduces to $\frac{1}{2}L_X\, g$, where L_X is the Lie derivative with respect to X.

We could look at the problem also as follows. The mathematical counterparts of the allowed (including reversible) deformations are the diffeomorphisms φ of a riemannian manifold (M, g), and every $\varphi(M)$ represents a possible configuration of the material considered. But some diffemorphisms do not lead to deformation (i.e. to shape changes), so, a criterion must be introduced to separate those diffeomorphisms which should be considered as essential. For such a criterion is chosen the distance change: *if the distance between any two fixed points does not change during the action of the external force field, then we say that there is no deformation.* Now, every essential diffeomorphism φ must transform the metric g to some new metric $\varphi^* g$, such that $g \neq \varphi^* g$. The naturally arising tensor field $e = (\varphi^* g - g) \neq 0$ appears as a measure of the physical abilities of the material to withstand external force actions.

Since the external force is assumed to act locally and the material considered gets the corresponding to the external force field final configuration in a smooth way, i.e. passing smoothly through a family of allowed configurations, we need a localization of the above scheme, such that the isometry doffeomorphisms to be eliminated. This is done by means of introducing 1-parameter group $\varphi_t, t \in [a, b] \subset \mathbb{R}$ of local diffeomorphisms, so, $\varphi_a(M)$ and $\varphi_b(M)$ denote correspondingly the initial and final configurations. Now φ_t generates a family of metrics $\varphi_t^*\, g$, and a corresponding family of tensors e_t. According to the local analysis every local 1-parameter group of diffeomorphisms is generated by a vector field on M. Let the vector field X generates φ_t. Then the quantity

$$\frac{1}{2}\, L_X g := \frac{1}{2} \lim_{t\to 0} \frac{\varphi_t^*\, g - g}{t} \ ,$$

i.e. one half of the *Lie derivative* of g along X, is called (infinitesimal) *strain tensor*, or *deformation tensor*.

Remark. Further in the paper we shall work with $L_X\, g$, i.e. the factor 1/2 will be omitted.

In our further study we shall call $L_X\, g$, where $g = \eta$ is the Minkowski (pseudo)metric, just *strain tensor*. We would like to note that, as far as we know, photon-like objects have not been considered from such a point of view. Clearly, the term "material" is not appropriate for PhLO because no static situations are admissible, **our objects of interest are of entirely dynamical nature**, so the corresponding *relativistic strain* tensors must take care of this.

According to the previous section important vector fields in our approach to describe electromagnetic PhLO are $\bar{\zeta}$, $\bar{A}$, $\bar{A}^*$, so, we consider the corresponding three electromagnetic strain tensors: $L_{\bar{\zeta}}\, \eta$; $L_{\bar{A}}\, \eta$; $L_{\bar{A}^*}\, \eta$.

Proposition 4. The following relations hold:

$$L_{\bar{\zeta}}\, \eta = 0, \quad (L_{\bar{A}}\, \eta)_{\mu\nu} \equiv D_{\mu\nu} = \left\| \begin{matrix} 2u_x & u_y + p_x & u_z & u_\xi \\ u_y + p_x & 2p_y & p_z & p_\xi \\ u_z & p_z & 0 & 0 \\ u_\xi & p_\xi & 0 & 0 \end{matrix} \right\| ,$$

$$(L_{\bar{A}^*}\, \eta)_{\mu\nu} \equiv D^*_{\mu\nu} = \left\| \begin{matrix} -2\varepsilon p_x & -\varepsilon(p_y + u_x) & -\varepsilon p_z & -\varepsilon p_\xi \\ -\varepsilon(p_y + u_x) & 2\varepsilon u_y & \varepsilon u_z & \varepsilon u_\xi \\ -\varepsilon p_z & \varepsilon u_z & 0 & 0 \\ -\varepsilon p_\xi & \varepsilon u_\xi & 0 & 0 \end{matrix} \right\| .$$

Proof. Immediately verified.

We give now some important from our viewpoint relations.

$$D(\bar{\zeta}, \bar{\zeta}) = D^*(\bar{\zeta}, \bar{\zeta}) = 0,$$

$$D(\bar\zeta)\equiv D(\bar\zeta)_\mu dx^\mu \equiv D_{\mu\nu}\bar\zeta^\nu dx^\mu=(u_\xi-\varepsilon u_z)dx+(p_\xi-\varepsilon p_z)dy,$$
$$D(\bar\zeta)^\mu\frac{\partial}{\partial x^\mu}\equiv D^\mu_\nu\bar\zeta^\nu\frac{\partial}{\partial x^\mu}=-(u_\xi-\varepsilon u_z)\frac{\partial}{\partial x}-(p_\xi-\varepsilon p_z)\frac{\partial}{\partial y}=-[\bar A,\bar\zeta],$$
$$D_{\mu\nu}\bar A^\mu\bar\zeta^\nu=-\frac{1}{2}\Big[(u^2+p^2)_\xi-\varepsilon(u^2+p^2)_z\Big]=-\frac{1}{2}L_{\bar\zeta}\phi^2,$$
$$D_{\mu\nu}\bar A^{*\mu}\bar\zeta^\nu=-\varepsilon\Big[u(p_\xi-\varepsilon p_z)-p(u_\xi-\varepsilon u_z)\Big]=-\varepsilon\mathbf{R}=-\varepsilon\phi^2\,L_{\bar\zeta}\psi.$$

We also have:

$$D^*(\bar\zeta)=\varepsilon\Big[-(p_\xi-\varepsilon p_z)dx+(u_\xi-\varepsilon u_z)dy\Big],$$
$$D^*(\bar\zeta)^\mu\frac{\partial}{\partial x^\mu}\equiv (D^*)^\mu_\nu\bar\zeta^\nu\frac{\partial}{\partial x^\mu}=-\varepsilon(p_\xi-\varepsilon p_z)\frac{\partial}{\partial x}+(u_\xi-\varepsilon u_z)\frac{\partial}{\partial y}=[\bar A^*,\bar\zeta],$$
$$D^*_{\mu\nu}\bar A^{*\mu}\bar\zeta^\nu=-\frac{1}{2}\Big[(u^2+p^2)_\xi-\varepsilon(u^2+p^2)_z\Big]=-\frac{1}{2}L_{\bar\zeta}\phi^2,$$
$$D^*_{\mu\nu}\bar A^{\mu}\bar\zeta^\nu=\varepsilon\Big[u(p_\xi-\varepsilon p_z)-p(u_\xi-\varepsilon u_z)\Big]=\varepsilon\mathbf{R}=\varepsilon\phi^2\,L_{\bar\zeta}\psi.$$

Clearly, $D(\bar\zeta)$ and $D^*(\bar\zeta)$ are linearly independent in general:

$$D(\bar\zeta)\wedge D^*(\bar\zeta)=\varepsilon\Big[(u_\xi-\varepsilon u_z)^2+(p_\xi-\varepsilon p_z)^2\Big]dx\wedge dy=\varepsilon\phi^2(\psi_\xi-\varepsilon\psi_z)^2\,dx\wedge dy\neq 0.$$

Recall now that every 2-form F defines a linear map $\tilde F$ from 1-forms to 3-forms through the exterior product: $\tilde F(\alpha):=\alpha\wedge F$, where $\alpha\in\Lambda^1(M)$. Moreover, the Hodge $*$-operator, composed now with $\tilde F$, gets $\tilde F(\alpha)$ back to $*\tilde F(\alpha)\in\Lambda^1(M)$. In the previous section we introduced two 2-forms $G=A\wedge\zeta$ and $G^*=A^*\wedge\zeta$ and noticed that $G^*=*G$. We readily obtain now

$$D(\bar\zeta)\wedge G=D^*(\bar\zeta)\wedge G^*=D(\bar\zeta)\wedge A\wedge\zeta=D^*(\bar\zeta)\wedge A^*\wedge\zeta=$$
$$=-\varepsilon\Big[u(p_\xi-\varepsilon p_z)-p(u_\xi-\varepsilon u_z)\Big]dx\wedge dy\wedge dz-\Big[u(p_\xi-\varepsilon p_z)-p(u_\xi-\varepsilon u_z)\Big]dx\wedge dy\wedge d\xi=$$
$$=-\phi^2\,L_{\bar\zeta}\psi\,(\varepsilon\,dx\wedge dy\wedge dz+dx\wedge dy\wedge d\xi)=-\mathbf{R}\,(\varepsilon\,dx\wedge dy\wedge dz+dx\wedge dy\wedge d\xi),$$
$$D(\bar\zeta)\wedge G^*=-D^*(\bar\zeta)\wedge G=D(\bar\zeta)\wedge A^*\wedge\zeta=-D^*(\bar\zeta)\wedge A\wedge\zeta=$$
$$=\frac{1}{2}\Big[(u^2+p^2)_\xi-\varepsilon(u^2+p^2)_z\Big](dx\wedge dy\wedge dz+\varepsilon\,dx\wedge dy\wedge d\xi).$$

Thus, recalling relations (12)-(15), we get

$$*\Big[D(\bar\zeta)\wedge A\wedge\zeta\Big]=*\Big[D^*(\bar\zeta)\wedge A^*\wedge\zeta\Big]=-\varepsilon\mathbf{R}\,\zeta=-i(\bar G^*)\mathbf{d}G=i(\bar G)\mathbf{d}G^*, \qquad (16)$$

$$*\Big[D(\bar\zeta)\wedge A^*\wedge\zeta\Big]=-*\Big[D^*(\bar\zeta)\wedge A\wedge\zeta\Big]=\frac{1}{2}L_{\bar\zeta}\phi^2\,\zeta=i(\bar G)\mathbf{d}G=i(\bar G^*)\mathbf{d}G^*. \qquad (17)$$

The above relations show various dynamical aspects of the energy-momentum redistribution during evolution of our PhLO. In particular, equations (16-17) clearly show that it is possible the translational and rotational components of the energy-momentum redistribution to be represented in form depending on the ζ-directed strains $D(\zeta)$ and $D^*(\zeta)$. So, the local translational changes of the energy-momentum carried by the two vector components G and G^* of our PhLO are given by the two 1-forms $*\big[D(\bar\zeta)\wedge A^*\wedge\zeta\big]$ and $*\big[D^*(\bar\zeta)\wedge A\wedge\zeta\big]$) and the local rotational ones - by the 1-forms $*\big[D(\bar\zeta)\wedge A\wedge\zeta\big]$ and $*\big[D^*(\bar\zeta)\wedge A^*\wedge\zeta\big]$. In fact, the form $*\big[D(\bar\zeta)\wedge A\wedge\zeta\big]$ determines the strain that "leaves" the 2-plane defined by (A,ζ) and the form $*\big[D^*(\bar\zeta)\wedge A^*\wedge\zeta\big]$ determines the strain that "leaves" the 2-plane defined by (A^*,ζ). Since the PhLO is free, i.e. no energy-momentum is lost or gained, this means that the two (null-field) components

G and G^* exchange locally *equal* energy-momentum quantities: $i(\bar{G}^*)\mathbf{d}G = -i(\bar{G})\mathbf{d}G^*$. Moreover, the easily verified relation $G_{\mu\sigma}G^{\nu\sigma} = (G^*)_{\mu\sigma}(G^*)^{\nu\sigma}$ shows that the two components G and G^* carry the same stress-energy-momentum. Now, the local energy-momentum conservation law $\nabla_\nu\big[G_{\mu\sigma}\bar{G}^{\nu\sigma} + (G^*)_{\mu\sigma}(\bar{G}^*)^{\nu\sigma}\big] = 0$ requires $L_{\bar{\zeta}}\phi^2 = 0$, and the corresponding strain-fluxes become zero: $*\big[D^*(\bar{\zeta}) \wedge A \wedge \zeta\big] = 0$, $*\big[D(\bar{\zeta}) \wedge A^* \wedge \zeta\big] = 0$. On the other hand, only dynamical relation between the energy-momentum change and strain fluxes exists, so NO analog of the assumed in elasticity theory generalized Hooke law, (i.e. linear relation between the stress tensor and the strain tensor) seems to exist. This clearly goes along with the fully dynamical nature of PhLO, i.e. linear relations exist between the divergence terms of our stress tensor $\frac{1}{2}\big[-G_{\mu\sigma}\bar{G}^{\nu\sigma} - (G^*)_{\mu\sigma}(\bar{G}^*)^{\nu\sigma}\big]$ and the $\bar{\zeta}$-directed strain fluxes as given by equations (16)-(17).

6 The translational-rotational consistency

We begin this section with summarizing from physical viewpoint some of the results of the preceding two sections in the following

Corollary. *An electromagnetic PhLO has two subsystems, mathematically represented by the two 2-forms G and G^*, these two subsystems carry the same energy-momentum, and they are in a permanent dynamical equilibrium: each one gives locally to the other as much energy-momentum as it gains locally from it.*

This conclusion and the considerations in the preceding two sections allow to make explicit the mathematical representation of the PhLO translational-rotational structure. In fact, it is seen that the rotational component of propagation of our PhLO is dimensionally localized in a 2-plane, which in our consideration is parametrized by coordinates (x, y), and the translational component of propagation is of constant nature and evolves along $\bar{\zeta}$. Since our PhLO is of electromagnetic nature, the corresponding energy-momentum tensor should be represented by $\frac{1}{2}\big[-G_{\mu\sigma}\bar{G}^{\nu\sigma} - (G^*)_{\mu\sigma}(\bar{G}^*)^{\nu\sigma}\big]$. Now, the corresponding local energy-momentum conservation law $\nabla_\nu\big[G_{\mu\sigma}\bar{G}^{\nu\sigma} + (G^*)_{\mu\sigma}(\bar{G}^*)^{\nu\sigma}\big] = 0$ reduces to the dynamical equation $L_{\bar{\zeta}}\phi^2 = L_{\bar{\zeta}}(u^2+p^2) = 0$, which seems to be naturally accepted to represent the translational component of propagation.

In order to come to some appropriate dynamical picture of the rotational component of propagation we make the following consideration. Recall the two vector fields $\bar{A}$ and $\bar{A}^*$. Since $\bar{A} \wedge \bar{A}^* = -\varepsilon(u^2 + p^2)\,\partial_x \wedge \partial_y \neq 0$, then at all space-time points occupied by our PhLO we have the frame $\Sigma_1 = (\bar{A}, \bar{A}^*, \partial_z, \partial_\xi)$. On the other hand the vector fields $[\bar{A}, \bar{\zeta}]$ and $[\bar{A}^*, \bar{\zeta}]$ are also linearly independent in general:

$$[\bar{A}, \bar{\zeta}] \wedge [\bar{A}^*, \bar{\zeta}] = \varepsilon\big[(u_\xi - \varepsilon u_z)^2 + (p_\xi - \varepsilon p_z)^2\big]\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} = \varepsilon\phi^2(\psi_\xi - \varepsilon\psi_z)^2\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\,,$$

and we get the new frame $\Sigma_2 = ([\bar{A}, \bar{\zeta}], [\bar{A}^*, \bar{\zeta}], \partial_z, \partial_\xi)$, which is regular only if $\mathbf{R} \neq 0$. Hence, the rotational component of propagation transforms the frame Σ_1 to the frame Σ_2. We see that essentially, the 2-frame $(\bar{A}, \bar{A}^*)$ is transformed to the 2-frame $([\bar{A}, \bar{\zeta}], [\bar{A}^*, \bar{\zeta}])$, and these two 2-frames are tangent to the (x, y)-plane. So we get the linear map

$$\big([\bar{A}, \bar{\zeta}], [\bar{A}^*, \bar{\zeta}]\big) = \big(\bar{A}, \bar{A}^*\big)\begin{Vmatrix}\alpha & \beta \\ \gamma & \delta\end{Vmatrix}.$$

Solving this system with respect to $(\alpha, \beta, \gamma, \delta)$ we obtain

$$\begin{Vmatrix}\alpha & \beta \\ \gamma & \delta\end{Vmatrix} = \frac{1}{\phi^2}\begin{Vmatrix}-\frac{1}{2}L_{\bar{\zeta}}\phi^2 & \varepsilon\mathbf{R} \\ -\varepsilon\mathbf{R} & -\frac{1}{2}L_{\bar{\zeta}}\phi^2\end{Vmatrix} = -\frac{1}{2}\frac{L_{\bar{\zeta}}\phi^2}{\phi^2}\begin{Vmatrix}1 & 0 \\ 0 & 1\end{Vmatrix} + \varepsilon L_{\bar{\zeta}}\psi\begin{Vmatrix}0 & 1 \\ -1 & 0\end{Vmatrix}.$$

Assuming the conservation law $L_{\bar\zeta}\phi^2=0$, we obtain that the rotational component of propagation is governed by the matrix $\varepsilon L_{\bar\zeta}\psi\,J$, where J denotes the canonical complex structure in $\mathbb{R}^2$, and since $\phi^2\,L_{\bar\zeta}\psi=\mathbf{R}$ we conclude that the rotational component of propagation is available if and only if the Frobenius curvature is NOT zero: $\mathbf{R}\neq 0$. We may also say that a consistent translational-rotational dynamical structure is available if the amplitude $\phi^2=u^2+p^2$ is a running wave along $\bar\zeta$ and the phase $\psi=\mathrm{arctg}\frac{p}{u}$ is NOT a running wave along $\bar\zeta$.

As we noted before the local conservation law $L_{\bar\zeta}\phi^2=0$, being equivalent to $L_{\bar\zeta}\phi=0$, gives one dynamical linear first order equation. This equation pays due respect to the assumption that our spatially finite PhLO, together with its energy density, propagates translationally with the constant velocity c. We need one more equation in order to specify the phase function ψ. If we pay corresponding respect also to the rotational aspect of the PhLO nature it is desirable this equation *to introduce and guarantee the conservative and constant character of this aspect of PhLO nature*. Since rotation is available only if $L_{\bar\zeta}\psi\neq 0$, the simplest such assumption respecting the constant character of the rotational component of propagation seems to be $L_{\bar\zeta}\psi=const\neq 0$. Now, since the usual physical dimension of the canonical coordinates (x,y,z,ξ) is [length] and the phase ψ is dimensionless, we may put $L_{\bar\zeta}\psi=const=\kappa/l_o$, where $\kappa=\pm 1$ and l_o is a positive constant with $[l_o]$=[length]. Note that l_o is equal to the square root of the relation of the volumes defined by the frames Σ_2 and Σ_1 : $l_o=\sqrt{|vol(\Sigma_2)/vol(\Sigma_1)|}=\sqrt{|\omega_o(\Sigma_2)/\omega_o(\Sigma_1)|}$.

Thus, the equation $L_{\bar\zeta}\phi=0$ and the frame rotation $[\bar A,\bar\zeta]=-\varepsilon\bar A^*\,L_{\bar\zeta}\psi$ and $[\bar A^*,\bar\zeta]=\varepsilon\bar A\,L_{\bar\zeta}\psi$, i.e. $(\bar A,\bar A^*,\partial_z,\partial_\xi)\rightarrow([\bar A,\bar\zeta],[\bar A^*,\bar\zeta],\partial_z,\partial_\xi)$, give the following equations for the two functions (u,p):

$$u_\xi-\varepsilon u_z=-\frac{\kappa}{l_o}\,p,\quad p_\xi-\varepsilon p_z=\frac{\kappa}{l_o}\,u\ .$$

If we now introduce the complex valued function $\Psi=u\,I+p\,J$, where I is the identity map in $\mathbb{R}^2$, the above two equations are equivalent to

$$L_{\bar\zeta}\Psi=\frac{\kappa}{l_o}J(\Psi)\ ,$$

which clearly confirms once again the translational-rotational consistency in the form that *no translation is possible without rotation, and no rotation is possible without translation*, where the rotation is represented by the complex structure J. Since the operator J rotates to angle $\alpha=\pi/2$, the parameter l_o determines the corresponding translational advancement, and $\kappa=\pm 1$ takes care of the left/right orientation of the rotation. Clearly, a full rotation (i.e. 2π-rotation) will require a $4l_o$-translational advancement, so, the natural time-period is $T=4l_o/c=1/\nu$, and $4l_o$ is naturally interpreted as the PhLO size along the spatial direction of translational propagation.

In order to find an integral characteristic of the PhLO rotational nature in *action units* we correspondingly modify, (i.e. multiply by l_o/c) and consider any of the two equal Frobenius curvature generating 4-forms:

$$\frac{l_o}{c}\,\mathbf{d}A\wedge A\wedge\zeta=\frac{l_o}{c}\,\mathbf{d}A^*\wedge A^*\wedge\zeta=\frac{l_o}{c}\,\varepsilon\mathbf{R}\omega_o=\frac{l_o}{c}\,\varepsilon\phi^2\,L_{\bar\zeta}\psi\,\omega_o=\varepsilon\kappa\frac{l_o}{c}\frac{\phi^2}{l_o}\,\omega_o=\varepsilon\kappa\frac{\phi^2}{c}\,\omega_o.$$

Integrating this 4-form over the 4-volume $\mathbb{R}^3\times[0,4l_o]$ we obtain the quantity $\mathcal{H}=\varepsilon\kappa ET=\pm\,ET$, where E is the integral energy of the PhLO and $T=4l_o/c$, which clearly is the analog of the Planck formula $E=h\nu$, i.e. $h=ET$.

7 Lagrangian formulation: a complex valued scalar field

Consider the space $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$, where $\mathbb{R}$ determines the time-direction. Let the field of complex numbers $\mathbb{C} = (\mathbb{R}^2, J),\ J \circ J = -id_{\mathbb{R}^2}$ be given a real representation as a 2-dimensional real vector space with basis

$$I = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad J = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}.$$

Every $\mathbb{C}$-valued function α on $\mathbb{R}^4$ can be represented in the form $\alpha = uI + pJ = \phi\cos\psi\, I + \phi\sin\psi\, J$, where u and p are two real-valued functions, $\phi = \sqrt{u^2+p^2}$ and $\psi = \arctan\frac{p}{u}$, and the components of α with respect to this basis will be numbered by latin indices taking values $(1,2) : \alpha_i, i = 1,2$. We denote further $J(\alpha) = -pI + uJ \equiv \bar{\alpha}$ and we shall make use of the introduced in the preceding section constant $l_o > 0$. Now we introduce two new 1-forms: the first one, denoted by k^s, is the restriction of $\frac{1}{l_o}\zeta$ to $\mathbb{R}^3$ and then extended to the whole $\mathbb{R}^4$ through zero-components in the $\bar{\zeta}$-adapted coordinate system: $k^s = \frac{\varepsilon}{l_o}dz$; the second one, denoted by k^ξ, is the restriction of $\frac{1}{l_o}\zeta$ to the time-direction $\mathbb{R}$ and then extended to the whole $\mathbb{R}^4$ in the same way: $k^\xi = \frac{1}{l_o}d\xi$. Hence, in the $\bar{\zeta}$-adapted coordinate system we obtain $k^s = (0,0,\varepsilon/l_0,0)$, and $k^\xi = (0,0,0,1/l_o)$.

Making use of our vector field $\bar{\zeta}$, of the inner product g in $\mathbb{C} = (\mathbb{R}^2, J)$ defined by $g(\alpha,\beta) = \frac{1}{2}tr(\alpha \circ \beta^*)$, β^* is the transposed to β, we consider now the following lagrangian (summation over the repeated indices: $g(\alpha,\alpha) = \alpha_i\alpha_i$):

$$\mathbb{L} = \frac{1}{2}\Big[\kappa l_o g(\alpha, L_{\bar{\zeta}}\bar{\alpha}) + g(\alpha,\alpha) - \kappa l_o g(\bar{\alpha}, L_{\bar{\zeta}}\alpha) + g(\bar{\alpha},\bar{\alpha})\Big] =$$

$$= \frac{1}{2}\Big[\alpha_i\Big(\kappa l_o \bar{\zeta}^\sigma \frac{\partial\bar{\alpha}_i}{\partial x^\sigma} + \alpha_i\Big) - \bar{\alpha}_i\Big(\kappa l_o \bar{\zeta}^\sigma \frac{\partial\alpha_i}{\partial x^\sigma} - \bar{\alpha}_i\Big)\Big],$$

where $\kappa = \pm 1$. Considering α and $\bar{\alpha}$ as independent, the Lagrange equations read

$$\kappa l_o \bar{\zeta}^\sigma \frac{\partial\bar{\alpha}_i}{\partial x^\sigma} = -\alpha_i \ ; \quad \kappa l_o \bar{\zeta}^\sigma \frac{\partial\alpha_i}{\partial x^\sigma} = \bar{\alpha}_i.$$

Note that the first equation follows from the second one under the action with J from the left, hence, we have just one equation of the form $\kappa l_o L_{\bar{\zeta}}\alpha = J(\alpha)$, which represents the idea for *consistent translational-rotational propagation*: the translational change $L_{\bar{\zeta}}\alpha$ of the field α is proportional to the rotational change $J(\alpha)$ and the coefficient l_o gives the translational advancement for a rotation of $\pi/2$.

In terms of $\phi = \sqrt{u^2+p^2}$ and $\psi = \arctan\frac{p}{u}$ these equations give

$$L_{\bar{\zeta}}\phi.\cos(\psi) - \phi.\sin(\psi)\left(L_{\bar{\zeta}}\psi - \frac{\kappa}{l_o}\right) = 0, \quad L_{\bar{\zeta}}\phi.\sin(\psi) + \phi.\cos(\psi)\left(L_{\bar{\zeta}}\psi - \frac{\kappa}{l_o}\right) = 0.$$

These two equations are consistent only if

$$L_{\bar{\zeta}}\phi = 0, \quad L_{\bar{\zeta}}\psi = \frac{\kappa}{l_o}. \tag{18}$$

The solutions are:

$$\phi = \phi(x,y,\xi+\varepsilon z); \quad \psi_1 = -\varepsilon\frac{\kappa}{l_o}z + f(x,y,\xi+\varepsilon z); \quad \psi_2 = \frac{\kappa}{l_o}\xi + f(x,y,\xi+\varepsilon z), \tag{19}$$

where f is an arbitrary function. Assuming $f = const$ we see that

$$\psi_1 = -\varepsilon\frac{\kappa}{l_o}z + const = -\kappa\, k^s_\mu x^\mu + const \quad \text{and} \quad \psi_2 = \frac{\kappa}{l_o}\xi + const = \kappa\, k^\xi_\mu x^\mu + const$$

are the simplest possible solutions leading to non-zero curvature. We note that the spatial structure of the solution defined by ψ_1 is *phase dependent* while the spatial structure of the solution defined by ψ_2 is NOT phase dependent.

We note that this lagrangian leads to the obtained in the previous section *linear* equations for the components of α, which equations admit 3d-finite solutions of the kind

$$\alpha_1 = \phi\cos\psi; \quad \alpha_2 = \phi\sin\psi$$

with consistent translational-rotational behavior, where ϕ and ψ are given above, and ϕ is a spatially finite function.

It is easily seen that the lagrangian becomes zero on the solutions, and since this lagrangian does NOT depend on any space-time metric the corresponding Hilbert energy-momentum tensor is zero on the solutions. This special feature of the lagrangian requires to look for another procedure leading to corresponding conserved quantities. A good candidate seems to be $T^{\mu\nu} = \phi^2\bar{\zeta}^\mu\bar{\zeta}^\nu$. In fact, we obtain (in our coordinate system)

$$\nabla_\nu T^{\mu\nu} = \bar{\zeta}^\mu\nabla_\nu(\phi^2\bar{\zeta}^\nu) + \phi^2\bar{\zeta}^\nu\nabla_\nu\bar{\zeta}^\mu = \bar{\zeta}^\mu L_{\bar{\zeta}}(\phi^2) + \phi^2\bar{\zeta}^\nu\nabla_\nu\bar{\zeta}^\mu.$$

The first term on the right is equal to zero on the solutions and the second term is zero since the vector field $\bar{\zeta}$ is autoparallel, so, $\nabla_\nu T^{\mu\nu} = 0$.

8 Lagrangian formulation: exterior 2-form field

We show now that the same equations for the two functions (u,p), or (ϕ,ψ), can be obtained from a lagrangian defined in terms of a 2-form. Recall that the space $\Lambda^2(\mathbb{R}^4)$ of 2-forms on $\mathbb{R}^4$ is 6-dimensional and denote by $\mathcal{J}$ the complex structure in this space defined by: $\mathcal{J}_{16} = -\mathcal{J}_{25} = \mathcal{J}_{34} = -\mathcal{J}_{43} = \mathcal{J}_{52} = -\mathcal{J}_{61} = 1$, and all other components of $\mathcal{J}$ are zero in our $\bar{\zeta}$-adapted coordinate system, i.e. the only non-zero elements are the off-diagonal components, and they alternatively change from $(+1)$ (upper right angle) to (-1) (lower left angle).

We define now a representation ρ of the algebra $\mathbb{C}$ in the algebra $L_{\Lambda^2(\mathbb{R}^4)}$ of linear maps in the 2-forms on $\mathbb{R}^4$ by the relation

$$\rho(\alpha_\varepsilon) = \rho(uI + \varepsilon pJ) \stackrel{\text{def}}{=} u\mathcal{I} + \varepsilon p\mathcal{J}, \quad \mathcal{I} = id_{\Lambda^2(\mathbb{R}^4)}, \quad \alpha_\varepsilon \in \mathbb{C}, \quad \varepsilon = \pm 1. \tag{20}$$

Clearly, $\rho(\alpha+\beta) = \rho(\alpha) + \rho(\beta)$, $\rho(\alpha.\beta) = \rho(\alpha)\circ\rho(\beta)$, and if G is an arbitrary 2-form then $\rho(\alpha_\varepsilon).G = uG + \varepsilon p\mathcal{J}(G)$. Note that here and further in the text the couple (u,p) may denote the complex number $(uI + pJ)$, as well as the complex-valued function $\alpha = u(x,y,z,\xi)I + p(x,y,z,\xi)J$.

Let's go back now to our $\bar{\zeta}$-adapted coordinate system and consider the 2-form $F_o = dx\wedge\zeta = dx\wedge(\varepsilon dz + d\xi) = \varepsilon dx\wedge dz + dx\wedge d\xi$. Recalling the two 1-forms $A = udx + pdy$ and $A^* = -pdx + udy$ (we omit ε before p and u as it was defined in Sec.2) we obtain

$$\rho(\alpha_\varepsilon).F_o = \varepsilon udx\wedge dz + \varepsilon pdy\wedge dz + udx\wedge d\xi + pdy\wedge d\xi = A\wedge\zeta,$$

$$\rho(J(\alpha_\varepsilon)).F_o = (-\varepsilon p\mathcal{I} + u\mathcal{J}).F_o =$$
$$= -pdx\wedge dz + udy\wedge dz - \varepsilon pdx\wedge d\xi + \varepsilon udy\wedge d\xi = A^*\wedge\zeta = \mathcal{J}(\rho(\alpha_\varepsilon).F_o).$$

Since ρ is a linear map and $\rho(0) = 0$, we get one-to-one map between the $\mathbb{C}$-valued functions on $\mathbb{R}^4$ and a special subset of 2-forms. All such 2-forms depend on the choice of the 1-form ζ, while the dependence on dx is not essential. Also, they are isotropic:

$$(A\wedge\zeta)\wedge(A\wedge\zeta) = 0, \ (A\wedge\zeta)\wedge\mathcal{J}(A\wedge\zeta) = (A\wedge\zeta)\wedge(A^*\wedge\zeta) = 0,$$

i.e. they have zero invariants.

Moreover, every such 2-form may be considered as a linear map in $\Lambda^2(\mathbb{R}^4)$ through the above correspondence: $\rho(\alpha_\varepsilon).F_o \rightarrow \rho(\alpha_\varepsilon)$. Since together with the zero element of $\Lambda^2(M)$ these 2-forms define a linear space V_ζ, this property suggests to introduce inner product in this linear space by the rule

$$< G^1_\varepsilon(a,b), G^2_\varepsilon(m,n) >= \frac{1}{6} tr\Big[\rho\big[\alpha_\varepsilon(m,n)\big] \circ \rho\big[\alpha^*_\varepsilon(a,b)\big]\Big] = am + bn.$$

Hence, every such 2-form acquires a norm.

Let now F and G be two arbitrary 2-forms. In order to define the lagrangian we consider the Minkowski space-time $M = (\mathbb{R}^4, \eta)$ as a real manifold, where the pseudoeuclidean metric η has signature $(-,-,-,+)$, and make use of the Lie derivative $L_{\bar\zeta}$ with respect to the vector field $\bar\zeta$. Also, $-(k^s)^2 = (k^\xi)^2 = (l_o)^{-2}$. Consider the lagrangian

$$\mathbb{L} = \eta\left(\kappa l_o L_{\bar\zeta} G + F, F\right) - \eta\left(\kappa l_o L_{\bar\zeta} F - G, G\right).$$

In components in the ζ-adapted coordinates where $\zeta^\sigma = const$, we can write

$$\mathbb{L} = \frac{1}{2}\left(\kappa l_o \bar\zeta^\sigma \frac{\partial G_{\alpha\beta}}{\partial x^\sigma} + F_{\alpha\beta}\right) F^{\alpha\beta} - \frac{1}{2}\left(\kappa l_o \bar\zeta^\sigma \frac{\partial F_{\alpha\beta}}{\partial x^\sigma} - G_{\alpha\beta}\right) G^{\alpha\beta}, \ \ 0 < l_o = const, \ \ \kappa = \pm 1, \quad (21)$$

where $F^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}$ and $G^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu}G_{\mu\nu}$. Note that this lagrangian is invariant with respect to $(F,G) \rightarrow (G,-F)$, or to $(F,G) \rightarrow (-G,F)$. The corresponding equations read

$$\kappa l_o \bar\zeta^\sigma \frac{\partial G_{\alpha\beta}}{\partial x^\sigma} + F_{\alpha\beta} = 0, \quad \kappa l_o \bar\zeta^\sigma \frac{\partial F_{\alpha\beta}}{\partial x^\sigma} - G_{\alpha\beta} = 0.$$

In coordinate-free form these equations look like

$$\kappa l_o L_{\bar\zeta} G = -F, \quad \kappa l_o L_{\bar\zeta} F = G.$$

Recall now the complex structure $\mathcal{J}$ and assume $F = -\mathcal{J}(G)$. Then, treating G and $\mathcal{J}(G)$ as independent (in fact they are linearly independent on the real manifold M), and in view of of the constancy of $\mathcal{J}$ in the ζ-adapted coordinates, we can write

$$\kappa l_o L_{\bar\zeta}\mathcal{J}(G) = -G, \quad \kappa l_o L_{\bar\zeta} G = \mathcal{J}(G). \qquad (22)$$

Since in our coordinates $\mathcal{J}$ and $\bar\zeta$ have constant coefficients, clearly, $L_{\bar\zeta}$ and $\mathcal{J}$ commute, so that the second (first) equation is obtained by acting with $\mathcal{J}$ from the left on the first (second) equation, i.e. the above mentioned invariance with respect to the transformations $(F,G) \rightarrow (\pm G, \mp F)$ is reduced to $\mathcal{J}$-invariance. Restricting now to 2-forms of the above defined kind and recalling the way how $\mathcal{J}$ acts, $\mathcal{J}(A \wedge \zeta) = A^* \wedge \zeta$, i.e. the couple (A,ζ) is rotated to the couple (A^*,ζ), we naturally interpret the last equations (22) as realization of the *translational-rotational consistency*: the translational change of G along $\bar\zeta$ is proportional to the rotational change of G determined by $\mathcal{J}$, so, roughly speaking, **no $\bar\zeta$-translation ($\mathcal{J}$-rotation) is possible without $\mathcal{J}$-rotation ($\bar\zeta$-translation)**, and the $\mathcal{J}$-rotation corresponds to l_o translational advancement.

From (22) it follows that on the solutions the lagrangian becomes zero. So, if we try to define the corresponding Hilbert energy-momentum tensor the variation of the volume element with respect to η is not essential. Moreover, the special quadratic dependence of $\mathbb{L}$ on η shows that the variation of $\mathbb{L}$ with respect to η will also become zero on the solutions. Hence, this is another example of the non-universality of the Hilbert method to define appropriate energy-momentum tensor. As for the canonical energy-momentum tensor, it is not symmetric, and its symmetrization is, in some extent, an arbitrary act, therefore, we shall not make use of it.

We continue to restrict the equations (22) onto the subset of 2-forms G of the kind $G = \rho(\alpha_\varepsilon).F_o$. As it was mentioned all these 2-forms have zero invariants: $G_{\mu\nu}G^{\mu\nu} = G_{\mu\nu}(\mathcal{J}(G))^{\mu\nu} = 0$, or in coordinate-free way, $G \wedge G = G \wedge \mathcal{J}(G) = 0$. Moreover, the easily verified relations $i(\bar{\zeta})G = i(\bar{\zeta})\mathcal{J}(G) = 0$ show an *intrinsic* connection to $\bar{\zeta}$: it is the only isotropic eigen vector of $G^\nu_\mu = \eta^{\mu\sigma}G_{\nu\sigma}$ and $(\mathcal{J}G)^\nu_\mu = \eta^{\mu\sigma}(\mathcal{J}G)_{\nu\sigma}$.

Substituting $G = \rho(\alpha_\varepsilon).F_o$ we get the already known equations

$$\kappa l_o(u_\xi - \varepsilon u_z) = -p, \quad \kappa l_o(p_\xi - \varepsilon p_z) = u \ . \tag{23}$$

Clearly, $\phi^2\zeta \otimes \bar{\zeta}$ is the right choice for energy-momentum tensor.

Note that the 2-form $F_o = dx \wedge \zeta$ satisfies the equation $L_{\bar{\zeta}}\phi = 0$ since $\phi_{F_o} = 1$, and does NOT satisfy the equation for ψ, since $\psi_{F_o} = 0, 2\pi, 4\pi, ...$, so, $L_{\bar{\zeta}}(\psi_{F_o}) = 0$. In view of this further we consider only not-constant $\mathbb{C}$-valued functions.

As we already mentioned an appropriate local representative of the rotational properties of these solutions appears to be any of the two Frobenius 4-forms $\mathbf{d}A \wedge A \wedge \zeta$ and $\mathbf{d}A^* \wedge A^* \wedge \zeta$, multiplied by the coefficient l_o/c, so that integrating over the 4-region $(\mathbb{R}^3 \times 4l_o)$ we get $\pm ET$, which carries integral information about the rotational properties of the solution.

We'd like to mention also that the 3-forms $i(\bar{\zeta})(\mathbf{d}A \wedge A \wedge \zeta) = i(\bar{\zeta})(\mathbf{d}A^* \wedge A^* \wedge \zeta)$, which in our coordinate system look like $\gamma \wedge \zeta$ with $\gamma = -\phi^2(L_{\bar{\zeta}}\psi)\, dx \wedge dy$, are closed.

The linear character of the equations obtained sets the question if the superposition principle holds. In general, let the parameters κ, ε, l_o of the two solutions be different. Let now $F_1(\kappa_1, \varepsilon_1, l^1_o; u, p)$ and $F_2(\kappa_2, \varepsilon_2, l^2_o; m, n)$ be two solutions along the same direction defined by $\bar{\zeta}$, and ε of ζ is of course equal to ε_1 for the first solution, and equal to ε_2 for the second solution. We ask now whether the linear combination $c_1F_1 + c_2F_2$ with $c_1 = const, c_2 = const$ will be also a solution $F_3(\kappa_3, \varepsilon_3, l^3_o; c_1u + c_2m, c_1p + c_2n)$ along the same direction? In order this to happen the following equations must be consistent:

$$\kappa_1\varepsilon_1 l^1_o L_{\bar{\zeta}}u = -\varepsilon_1 p, \quad \kappa_1\varepsilon_1 l^1_o L_{\bar{\zeta}}p = \varepsilon_1 u, \quad \kappa_2 l^2_o L_{\bar{\zeta}}m = -\varepsilon_2 n, \quad \kappa_2 l^2_o L_{\bar{\zeta}}n = \varepsilon_2 m$$

$$\kappa_3\varepsilon_3 l^3_o L_{\bar{\zeta}}(c_1u + c_2m) = -\varepsilon_3(c_1p + c_2n), \quad \kappa_3\varepsilon_3 l^3_o L_{\bar{\zeta}}(c_1p + c_2n) = \varepsilon_3(c_1u + c_2m),$$

where ε_3 is equal ε_1, or to ε_2. The corresponding consistency condition looks as follows:

$$\kappa_3\varepsilon_3 l^3_o = \frac{c_1p + c_2n}{\frac{\varepsilon_1\kappa_1c_1}{l^1_o}p + \frac{\varepsilon_2\kappa_2c_2}{l^2_o}n} = \frac{c_1u + c_2m}{\frac{\varepsilon_1\kappa_1c_1}{l^1_o}u + \frac{\varepsilon_2\kappa_2c_2}{l^2_o}m}.$$

For example, the relations $\varepsilon_3\kappa_3 l^3_o = \varepsilon_2\kappa_2 l^2_o = \varepsilon_1\kappa_1 l^1_o$ are sufficient for this superposition to be a solution. This means that if the two solutions propagate translationally for example from $-\infty$ to $+\infty$, i.e. $\varepsilon_1 = \varepsilon_2 = -1$, if the rotational orientations coincide, i.e. $\kappa_1 = \kappa_2$, and if the spatial periodicity parameters are equal: $l^1_o = l^2_o = l^3_o$, the sum $(c_1F_1 + c_2F_2)$ gives a solution. In general, however, the combination $c_1F_1 + c_2F_2$ will not be a solution.

On the other hand, we can introduce a multiplicative structure in the solutions of the kind $\rho(\alpha_\varepsilon).F_o$. In fact, if $F_1(\kappa_1) = \rho(\alpha^1(\varepsilon_1)).F_o$ and $F_2(\kappa_2) = \rho(\alpha^2(\varepsilon_2)).F_o$ are two such solutions we define their product $F = F_1.F_2$ by $F = F_1.F_2 = \rho(\alpha^1.\alpha^2).F_o$. Clearly, the amplitude of F is a product of the amplitudes of F_1 and F_2: $\phi_F = \phi_{F_1}.\phi_{F_2}$ and the phase of F is the sum of the phases of F_1 and F_2: $\psi_F = \psi_{F_1} + \psi_{F_2}$. Now, F will be a solution only if

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_F, \quad \frac{\kappa_F}{l^F_o} = \frac{\kappa_1 l^2_o + \kappa_2 l^1_o}{l^1_o.l^2_o}.$$

From the first of these relations it follows that in order F to be a solution, F_1 and F_2 must NOT move against each other, and then the product-solution shall follow the same translational direction. However, it is allowed F_1 and F_2 to have different rotational orientations, i.e. $\kappa_1 \neq \kappa_2$,

then the product-solution will have rotational orientation $\kappa_F = sign(\kappa_1 l_o^2 + \kappa_2 l_o^1)$. Clearly, every subset of solutions with the same $(\varepsilon, \kappa, l_o)$ form a group with neutral element $F_o = \rho(I).F_o$, and $F_\alpha^{-1} = \rho(\alpha^{-1}).F_o$, moreover, the multiplicative group of $\mathbb{C}$ acts on these solutions: $(\alpha_\varepsilon, F_\varepsilon) \rightarrow \rho(\alpha_\varepsilon).F_\varepsilon$.

We give now an explicit solution, i.e. a procedure to construct a solution. We assume that the phase is given by ψ_1 and ε and κ take values ± 1 independently. The form of this solution $u = \phi \cos \psi_1;\;\; p = \phi \sin \psi_1$ shows that the initial condition is determined entirely by the choice of ϕ, and it suggests also to choose the initial condition $\phi_{t=0}(x, y, \varepsilon z)$ in the following way. Let for $z = 0$ the initial condition $\phi_{t=0}(x, y, 0)$ be located on a disk $D = D(x, y; a, b; r_o)$ of small radius r_o, the center of the disk to have coordinates (a, b), and the value of $\phi_{t=0}(x, y, 0)$ to be proportional to the distance $R(x, y, 0)$ between the origin of the coordinate system and the point $(x, y, 0)$, so, $R(x, y, 0) = \sqrt{x^2 + y^2}$, and D is defined by $D = \{(x, y)|\sqrt{(x-a)^2 + (y-b)^2} \leq r_o\}$. Also, let θ_D be the smoothed out characteristic function of the disk D, i.e. $\theta_D = 1$ everywhere on D except a very thin hoop-like zone $B_D \subset D$ close to the boundary of D where θ_D rapidly goes from 1 to zero (in a smooth way), and $\theta_D = 0$ outside D. Let also the dependence on z be given by be the corresponding characteristic function $\theta(z; 4l_o)$ of an interval $(z, z + 4l_o)$ of length $4l_o$ on the z-axis. If γ is the proportionality coefficient we obtain

$$\phi(x, y, z, ct + \varepsilon z) = \gamma.R(x, y, 0).\theta_D.\theta(ct + \varepsilon z; 4l_o).$$

We see that because of the available *sine* and *cosine* factors in the solution, the initial condition for the solution will occupy a helical cylinder of height $4l_o$, having internal radius of r_o and wrapped up around the z-axis. Also, its center will always be $R(a, b, 0)$-distant from the z-axis. Hence, the solution will propagate translationally along the coordinate z with the velocity c, and rotationally inside the corresponding infinitely long helical cylinder because of the z-dependence of the available periodical multiples. The curvature K and the torsion T of the screwline through the point $(x, y, 0) \in D$ will be

$$K = \frac{R(x, y, 0)}{R^2(x, y, 0) + b^2}, \qquad T = \frac{\kappa\, b}{R^2(x, y, 0) + b^2}\ ,$$

where $b = 2l_o/\pi$. The rotational frequency ν will be $\nu = c/2\pi b$, so we can introduce period $T = 1/\nu$ and elementary action $h = E.T$, where E is the (obviously finite) integral energy of the solution defined as 3d-integral of the energy density ϕ^2.

9 Discussion and Conclusion

The two basic features of our approach are the assumptions for continuous spatially finite structure, and for available consistent translational-rotational dynamical space-time structure of PhLO. Hence, PhLO *propagate*, they do not move. The spatial structure is of composite nature and has two interrelated components represented by the 2-forms G and G^*, and the propagation has two components of constant nature: *translational* and *rotational*. The translational component is along isotropic straight lines in $(\mathbb{R}^4, \eta)$ with constant speed c. The rotational component of propagation is also of constant nature and follows the special rotational properties of PhLO's spatial structure. This intrinsically consistent dual nature of PhLO demonstrates itself according to the rule: **no translation (rotation) is possible without rotation (translation)**.

While the translational component of propagation is easily accounted through the (arbitrary chosen in general) isotropic autoparallel vector field $\bar{\zeta}$, the rotational component of propagation was introduced in our model making use of two things: the non-integrability properties of the induced by $\bar{\zeta}$ two 2-dimensional differential/Pfaff systems, and the complex structure $\mathcal{J}$. This approach brought the following important consequences:

1. It automatically led to the required translational-rotational consistency.

2. The rotational properties of the solutions are *intrinsic* for the PhLO nature, they are transversal to $\bar{\zeta}$, they are in accordance with the action of the complex structure $\mathcal{J}$ on the 2-forms G and G^*, and they allow the characteristics *amplitude* ϕ and *phase* ψ of a solution to be correctly introduced. Moreover, the rotation is of periodical nature, and a helical spatial structure along the spatial direction of propagation is allowed, so, the corresponding rotational properties differ from those in the case of rotation of a solid as a whole around a point or axis.

3. The spatial shape and translational properties of a solution are carried by the amplitude ϕ : $L_{\bar{\zeta}}\phi = 0$, and the rotational ones are carried by the phase ψ: $L_{\bar{\zeta}}\psi \neq 0$.

4. The curvature, considered as a measure of the available non-integrability, or equivalently as a $(\bar{A}^*, \bar{\zeta})$-directed strain $D(\bar{A}^*, \bar{\zeta})$ (or as a $(\bar{A}, \bar{\zeta})$-directed strain $D^*(\bar{A}, \bar{\zeta})$) , is non-zero only if the phase ψ is NOT a running wave along $\bar{\zeta}$: $L_{\bar{\zeta}}\psi \neq 0$, hence, roughly speaking, curvature means rotation and vice versa.

5. Quantitatively, the curvature $\mathbf{R}$ is *obtained* to be proportional to the product of the energy-density ϕ^2 and the phase change $L_{\bar{\zeta}}\psi = const$ along $\bar{\zeta}$ of a solution : $\mathbf{R} = l_o\,\phi^2$. We recall that in General Relativity (GR) a proportionality relation between the energy-momentum density of *non-gravitational* fields and the correspondingly contracted riemannian curvature is *postulated*, while here it is *obtained* in the most general sense of the concept of curvature, namely, as a measure of Frobenius non-integrability (of correspondingly defined subdistributions of an integrable distribution which is meant to represent a PhLO). So, if we believe that the gravitational field carries energy-momentum, this result could be considered as a strong support for the GR ideology if the gravitational energy-momentum is manifestly included and treated in the same way as the energy-momentum characteristics of all other fields.

6. A natural *integral* measure of the rotational properties of a solution appears to be the product ET, i.e. the action for one period $T = 4l_o/c$, which is in accordance with the Planck formula $ET = h$.

Together with the allowed finite nature of the solutions these properties suggest the following understanding of the PhLO's time-stable dynamical structure: **PhLO MUST always propagate in a translational-rotational manner as fast as needed in order to "survive", i.e. to overcome the instability (the destroying tendencies), generated by the available non-integrability**. In other words, every free PhLO has to be able to supply immediately itself with those existence needs that are constantly put under the non-integrability destroying influence. Some initial steps to understand quantitatively this "smart" nature of PhLO in the terms used in the paper could be the following.

Recalling the two 2-forms $G = A \wedge \zeta$ and $G^* = A^* \wedge \zeta$ we see that when $L_{\bar{\zeta}}\phi^2 = 0$ then the corresponding subsystems keep the energy-momentum carried by each of them: $i(\bar{G})\mathbf{d}G = i(\bar{G}^*)\mathbf{d}G^* = 0$. On the other hand the relation $i(\bar{G})\mathbf{d}G^* = -i(\bar{G}^*)\mathbf{d}G = -\varepsilon\mathbf{R}.\zeta$ may be physically interpreted in two ways. FIRST, differentially, G transfers to G^* so much energy-momentum as G^* transfers back to G, which goes along with the previous relations stating that G and G^* keep their energy-momentum densities. Each of these two quantities $i(\bar{G})\mathbf{d}G^*$ and $i(\bar{G}^*)\mathbf{d}G$ is equal (up to a sign) to $\mathbf{R}.\zeta$, so, such mutual exchange of energy-momentum is possible only if the non-integrability of each of the two Pfaff 2-dimensional systems (A, ζ) and (A^*, ζ) is present, i.e. when the curvature $\mathbf{R}$ is NOT zero and is measured by the same non-zero quantity.

Since the curvature implies outside directed flow (with respect to the corresponding 2-dimensional distribution) this suggests the SECOND interpretation: the energy-momentum that PhLO might lose differentially in *whatever way* by means of G is differentially and simultaneously supplied by means of G^*, and vice versa. We could say that every PhLO has two functioning subsystems, G and G^*, such, that the energy-momentum loss through the subsystem G generated by the nonintegrability of (A, ζ), is gained (or supplied) back by the subsystem

G^*, and vice versa, and in doing this PhLO make use of the corresponding rotational component of propagation supported by appropriate spatial structure. All this is mathematically guaranteed by the isotropic character of G and G^*, i.e. by the zero values of the two invariants $G_{\mu\nu}G^{\mu\nu} = G^*_{\mu\nu}G^{\mu\nu} = 0$, and by making use of the complex structure $\mathcal{J}$ as a rotation generating operator.

Let's now try to express this dually consistent dynamical nature of PhLO by *one* object which satisfies *one* relation. We are going to consider G and G^* as two vector components of a $(\mathbb{R}^2, J)$-valued 2-form, namely, $\Omega = G \otimes I + G^* \otimes J$. Applying the exterior derivative we get $\mathbf{d}\Omega = \mathbf{d}G \otimes I + \mathbf{d}G^* \otimes J$. Consider now the $(\mathbb{R}^2, J)$-valued 2-vector $\bar{\Omega} = \bar{G} \otimes I + \bar{G^*} \otimes J$. The aim we pursue will be achieved through defining the object ($\vee$ is the symmetrized tensor product)

$$(\vee, i)(\bar{\Omega}, \mathbf{d}\Omega) \stackrel{\text{def}}{=} i(\bar{G})\mathbf{d}G \otimes I \vee I + i(\bar{G^*})\mathbf{d}G^* \otimes J \vee J + \Big[i(\bar{G})\mathbf{d}G^* + i(\bar{G^*})\mathbf{d}G\Big] \otimes I \vee J$$

and put it equal to zero: $(\vee, i)(\bar{\Omega}, \mathbf{d}\Omega) = 0$.

We note that this last relation $(\vee, i)(\bar{\Omega}, \mathbf{d}\Omega) = 0$ represents the dynamical equations of the vacuum Extended Electrodynamics (Donev, Tashkova 1995, 2004). In particular, this equation contains all solutions to the Maxwell vacuum equations, and the solutions obtained in this paper are a special part of the full subset of nonlinear solutions to these nonlinear equations.

From a general point of view it deserves to emphasize once again that the available curvature 2-forms of nonintegrable subdistributions of an integrable one, represent a natural instrumentarium for describing and understanding the structure of the complex of internal exchange processes, which processes guarantee the time-stability of a composit physical object/system having dynamical structure. The simple case of PhLO considered here suggests further applications of this approach to physical objects/systems of more complicated dynamical internal structure.

In conclusion, spatially finite field models of PhLO can be built in terms of complex valued functions and in terms of isotropic 2-forms on Minkowski space-time, and these two approaches can be related. In these both cases substantial role play the complex structures J and $\mathcal{J}$ as rotation generating operators, carrying information in this way about the spin properties of PhLO.

The PhLO's longitudinal sizes and rotational orientations can be determined by the constant parameter combination κl_o, and their effective transversal sizes may be considered to depend on the energy-density carried by them. The time-stability is guaranteed, on one hand, by the internal energy-momentum exchange between the two non-integrable 2-dimensional differential/Pfaff systems, and on the other hand, by a (possible) dynamical harmony with the outside world.

The significance of the concepts of Frobenius curvature and $\bar{\zeta}$-directed strain appear to be of primary importance in our approach.

The results presented in the paper clearly suggest that the Frobenius integrability theory appears to be important mathematical scheme to be used in theoretical physics when time-stable physical systems of composite structure, i.e. considered as built of relatively time-stable mutually interacting subsystems, are studied. The arizing Frobenius curvatures acquire natural interpretations of objects used to build the mathematical representatives of the corresponding energy-momentum exchanges, i.e. force fields. Interaction of the physical system with the outside world can also be incorporated in this scheme.

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